

# Partial sums of the Möbius function in arithmetic progressions assuming GRH

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## Abstract

We consider Mertens' function in arithmetic progression,

$$M(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu(n).$$

Assuming the generalized Riemann hypothesis (GRH), we show that the bound

$$M(x, q, a) \ll_{\varepsilon} \sqrt{x} \exp\left((\log x)^{3/5} (\log \log x)^{16/5+\varepsilon}\right)$$

holds uniform for all  $q \leq \exp\left(\frac{\log 2}{2} \left\lfloor (\log x)^{3/5} (\log \log x)^{11/5} \right\rfloor\right)$ ,  $\gcd(a, q) = 1$  and all  $\varepsilon > 0$ . The implicit constant is depending only on  $\varepsilon$ . For the proof, a former method of K. Soundararajan is extended to  $L$ -series.

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# 1 Introduction

Mertens' function is defined by

$$M(x) := \sum_{n \leq x} \mu(n).$$

It is well known that  $M(x) = o(x^{1/2+\varepsilon})$  is equivalent to Riemann's hypothesis.

When assuming Riemann's hypothesis for  $\zeta$ , one can give even sharper bounds for  $M(x)$ , see [3], [8], [4], [7], [5]:

In [4], H. Maier and H. L. Montgomery proved the bound

$$M(x) \ll x^{1/2} \exp\left(c(\log x)^{39/61}\right) \text{ for a } c > 0.$$

In [7], K. Soundararajan improved the bound by showing

$$M(x) \ll x^{1/2} \exp\left((\log x)^{1/2} (\log \log x)^{14}\right).$$

In [5], A. de Roton und M. Balazard refine the result of K. Soundararajan and show

$$M(x) \ll_{\varepsilon} \sqrt{x} \exp\left((\log x)^{1/2} (\log \log x)^{5/2+\varepsilon}\right),$$

which is the best bound up to date.

In this paper we generalize the method of K. Soundararajan to provide a bound for Mertens' function in arithmetic progression,

$$M(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu(n).$$

Note that the trivial bound is  $\leq x/q$ , so bounds smaller than  $x^{1/2+\varepsilon}$  are nontrivial if  $q \leq x^{1/2-\varepsilon}$ .

We adapt the method of K. Soundararajan resp. the modification of A. de Roton and M. Balazard in such a way, that it remains applicable for Dirichlet  $L$ -series. We obtain the following nontrivial upper bound assuming Riemann's hypothesis for all Dirichlet  $L$ -series  $L(s, \chi)$  with  $\chi \pmod{q}$  and all moduli  $q$  in question (GRH for short):

**Theorem 1.** *Assuming GRH, the bound*

$$M(x, q, a) \ll_{\varepsilon} \sqrt{x} \exp\left((\log x)^{3/5} (\log \log x)^{16/5+\varepsilon}\right)$$

*holds uniform for all  $q \leq \exp\left(\frac{\log 2}{2} \left\lfloor (\log x)^{3/5} (\log \log x)^{11/5} \right\rfloor\right)$ ,  $\gcd(a, q) = 1$  and all  $\varepsilon > 0$  with an implicit constant depending only on  $\varepsilon$ .*

With this theorem, we extend the results of [7] resp. [5] to a Siegel-Walfisz-type result. The obtained bound is weaker than the one of [7] resp. [5], but still sharper than the one of [4].

The method we use is as follows. We expand the Möbius sum  $M(x, q, a)$  using Dirichlet characters,

$$\begin{aligned} M(x, q, a) &= \frac{1}{\varphi(q)} \sum_{\chi(q)} \bar{\chi}(a) \sum_{n \leq x} \chi(n) \mu(n) \\ &= \frac{1}{\varphi(q)} \sum_{\chi(q)} \bar{\chi}(a) A(x, \chi, q) + O(\log x), \end{aligned}$$

using Perron's formula with integrals

$$A(x, q, \chi) = \frac{1}{2\pi i} \int_{1+1/(\log x)-i2^K}^{1+1/(\log x)+i2^K} \frac{x^s}{L(s, \chi)s} ds, \quad K := \left\lfloor \frac{\log x}{\log 2} \right\rfloor.$$

With this, bounds for  $L(s, \chi)$  are needed. Considering the principal character  $\chi_0 \bmod q$ , the formula

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right)$$

shows that already the sharper bound of [7]/[5] applies (see the proof of Lemma 3). So the main work is to consider nonprincipal characters.

Like in [7]/[5], the main steps are then some propositions aiming to bound  $L(s, \chi)$  to obtain an upper estimate for  $A(x, q, \chi)$ . They are given in Sections 7 and 8 and are resulting from the propositions in the former Sections 2 and 4,5,6.

Most of these propositions are stated for primitive characters. If necessary, results for nonprimitive characters  $\chi \neq \chi_0$  are derived by reduction to a primitive character that induces  $\chi$ .

The main idea in [7], namely the concept of  $V$ -typical ordinates, is extended to a version which allows one to work also with  $L$ -series. We give the adapted definition in Section 3.

As one important step, we show in Section 4 that there *are* actually  $V$ -typical ordinates, see Proposition 8.

In Section 5, it is shown that short intervals containing an unusual number of ordinates of nontrivial  $L$ -zeros  $\bmod q$  do not appear too often, even uniformly for all  $q$  up to the given bound (Proposition 9), so the  $V$ -untypical ordinates are small in number (Proposition 10). In the case of  $\zeta$ , this has been the breakthrough in Soundararajan's paper [7].

The resulting bound and the range for  $q$  in Theorem 1 is then obtained by optimizing the bounds for  $A(x, q, \chi)$  in Section 9. The elementary Proposition 20 plays an intrinsic rôle for this.

A remark on notations used in this paper:

We mark all Propositions that assume the generalized Riemann hypothesis by the symbol (GRH). We stress that all implicit constants are absolute unless otherwise indicated.

## 2 List of tools

In this section, we give a collection of the tools used in the proof.

The first proposition gives an approximation of the characteristic function of a given interval:

**Proposition 1.** *Let  $h > 0$ ,  $\Delta \geq 1$ .*

*Let  $\mathbf{1}_{[-h, h]}$  be the characteristic function of the interval  $[-h, h]$ .*

*There are even, entire functions  $F_+$  and  $F_-$  depending on  $h$  and  $\Delta$ , being real on the real axis and such that the following properties hold:*

1.  $\forall u \in \mathbb{R} : F_-(u) \leq \mathbf{1}_{[-h, h]}(u) \leq F_+(u),$
2.  $\int_{-\infty}^{\infty} |F_{\pm}(u) - \mathbf{1}_{[-h, h]}(u)| du = 1/\Delta$  and  $\hat{F}_{\pm}(0) = 2h \pm 1/\Delta,$

3.  $\hat{F}_\pm$  is realvalued and even, and we have  $\hat{F}_\pm(x) = 0$  for all  $|x| \geq \Delta$  and  $|x\hat{F}_\pm(x)| \leq 2$  for all  $x \in \mathbb{R}$ ,  
4. for  $z \in \mathbb{C}$  with  $|z| \geq \max\{2h, 1\}$  we have

$$|F_\pm(z)| \ll \frac{\exp(2\pi|\Im z|\Delta)}{(\Delta|z|)^2}.$$

The proof uses Beurling's Approximation of the signum function

$$\text{sgn}(x) := \begin{cases} x/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Let  $K(z) := \left(\frac{\sin(\pi z)}{\pi z}\right)^2$  and  $H(z) = K(z) \left(\sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n)}{(z-n)^2} + \frac{2}{z}\right)$ , then it can be shown that the functions

$$F_\pm(z) := \frac{1}{2}(H(\Delta(z+h)) \pm K(\Delta(z+h)) + H(\Delta(h-z)) \pm K(\Delta(h-z)))$$

have the properties asserted in Proposition 1. This can be seen as in [6] and [9], see also [5], we just give the proof of part 4. in more detail:

For this, let  $z = x + iy$  with  $x, y \in \mathbb{R}$  and  $|z| \geq \max\{2h, 1\}$ . Since  $F_\pm$  are even, consider only nonnegative  $x$ . Using  $\sin(z) \ll e^{|\Im(z)|}$  and  $\Im(\Delta(z+h)) = -\Im(\Delta(h-z)) = \Delta\Im(z)$ , we get the desired bound for  $K(\Delta(z+h)) \pm K(\Delta(h-z))$  since  $|z \pm h| = |z| \left|1 \pm \frac{h}{z}\right| \geq |z| \left(1 - \frac{h}{|z|}\right) \geq \frac{1}{2}|z|$ .

To estimate  $H(\Delta(z+h)) + H(\Delta(h-z))$  we use the identities

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}, \text{ converging on every compact subset of } \mathbb{C} \setminus \mathbb{Z}, \quad (1)$$

$$\sum_{n=0}^{\infty} \frac{1}{(z+n)(z+n+1)} = \frac{1}{z}, \text{ converging absolutely for } z \in \mathbb{C} \setminus -\mathbb{N}_0. \quad (2)$$

Consider  $H(\Delta(z+h))$  and  $H(\Delta(h-z))$  separately. By (1), we have

$$\begin{aligned} H(\Delta(z+h)) &= \left(\frac{\sin(\pi\Delta(z+h))}{\pi}\right)^2 \left(\sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n)}{(\Delta(z+h)-n)^2} + \frac{2}{\Delta(z+h)}\right) \\ &= 1 + \left(\frac{\sin(\pi\Delta(z+h))}{\pi}\right)^2 \left(-2 \sum_{n=1}^{\infty} \frac{1}{(\Delta(z+h)+n)^2} - \frac{1}{(\Delta(z+h))^2} + \frac{2}{\Delta(z+h)}\right), \end{aligned}$$

and (2) gives for the negative of the last term in large brackets the expression

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\frac{1}{(\Delta(z+h)+n)^2} + \frac{1}{(\Delta(z+h)+n+1)^2}\right) - \sum_{n=0}^{\infty} \frac{2}{(\Delta(z+h)+n)(\Delta(z+h)+n+1)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{(\Delta(z+h)+n)} - \frac{1}{(\Delta(z+h)+n+1)}\right)^2 = \sum_{n=0}^{\infty} \frac{1}{(\Delta(z+h)+n)^2(\Delta(z+h)+n+1)^2} \\ &\leq \frac{1}{(\Delta(x+h+|y|))^2} \sum_{n=0}^{\infty} \frac{1}{(\Delta(x+h+|y|)+n)(\Delta(x+h+|y|)+n+1)} \end{aligned}$$

$$= \frac{1}{(\Delta(x+h+|y|))^3} \ll \frac{1}{|\Delta(z+h)|^3} \ll \frac{1}{|\Delta z|^3}.$$

Analogously, we get

$$\begin{aligned} H(\Delta(h-z)) &= \left( \frac{\sin(\pi\Delta(h-z))}{\pi} \right)^2 \left( \sum_{-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(\Delta(h-z)-n)^2} + \frac{2}{\Delta(h-z)} \right) \\ &= -1 + \left( \frac{\sin(\pi\Delta(z-h))}{\pi} \right)^2 \left( \frac{1}{(\Delta(z-h))^2} + 2 \sum_1^{\infty} \frac{1}{(\Delta(z-h)+n)^2} - \frac{2}{\Delta(z-h)} \right). \end{aligned}$$

If  $\Re(z) > h$ , the treatment of the last term in large brackets is as before.

So let  $\Re(z) \leq h$ . Due to  $|z| \geq 2h$ , we have  $|y| = |\Im(z)| > h$ , so  $z \notin \mathbb{R}$  and  $|\Im(z)| \geq |\Re(z)|$ . Again (2) gives for the last term in large brackets the expression

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{(\Delta(z-h)+n)^2(\Delta(z-h)+n+1)^2} \\ &\ll \sum_{0 \leq n \leq \Delta h} \frac{1}{|(\Delta(x-h)+n|+\Delta|y|)^2(|\Delta(x-h)+n+1|+\Delta|y|)^2} \\ &+ \sum_{n > \Delta h} \frac{1}{|(\Delta(x-h)+n|+\Delta|y|)^2(|\Delta(x-h)+n+1|+\Delta|y|)^2} \\ &\ll \frac{\max\{\Delta h, 1\}}{|\Delta y|^4} + \sum_{n=0}^{\infty} \frac{1}{(\Delta|y|+n)^2(\Delta|y|+n+1)^2} \ll \frac{1}{|\Delta y|^3} \ll \frac{1}{|\Delta z|^3}. \end{aligned}$$

Summing up we obtain

$$H(\Delta(z+h)) + H(\Delta(h-z)) \ll \frac{e^{2\pi\Delta|\Im(z)|}}{(\Delta|z|)^3}$$

and the desired bound for  $|z| \geq \max\{2h, 1\}$ . □

We will make use of the following explicit formula for the functions  $F_{\pm}$ .

**Proposition 2.** (GRH) Let  $\chi$  be a primitive character mod  $q$ . Let  $t > 0$ ,  $\Delta \geq 1$ ,  $h > 0$ , and  $F_{\pm}$  the functions from Proposition 1. Then we have

$$\begin{aligned} \sum_{\rho=\frac{1}{2}+i\gamma} F_{\pm}(\gamma-t) &= \frac{1}{2\pi} \hat{F}_{\pm}(0) \log \frac{q}{\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\pm}(u-t) \Re \frac{\Gamma'}{\Gamma} \left( \frac{\frac{1}{2}+iu+\mathfrak{a}}{2} \right) du \\ &\quad - \frac{1}{\pi} \Re \sum_{n \in \mathbb{N}} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \hat{F}_{\pm} \left( \frac{\log n}{2\pi} \right). \end{aligned}$$

Here the sum on the left hand side runs through all zeros of  $L(s, \chi)$  in the strip  $0 \leq \sigma \leq 1$  with relevant multiplicity, and where we have set

$$\mathfrak{a} := \mathfrak{a}(\chi) := \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1. \end{cases} \quad (3)$$

The proof can be established in the same way as Theorem 5.12, p. 108, in the book [2] of Iwaniec and Kowalski. It uses the Mellin transform, the explicit formula for  $\frac{L'}{L}(s, \chi)$  and the residue theorem, where one has to take care of the trivial zero of  $L(s, \chi)$  at  $s = 0$  if  $\chi(-1) = 1$ .

An estimate of the integral in Proposition 2 gives the next proposition:

**Proposition 3.** *Let  $t \geq 25$ ,  $\Delta \geq 1$ ,  $0 < h \leq \sqrt{t}$ ,  $F_{\pm}$  as in Proposition 1,  $\chi$  a character mod  $q$ . Then it holds that*

$$\int_{-\infty}^{\infty} F_{\pm}(u-t) \Re \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{\mathfrak{a} + it}{2} \right) du = \left( 2h \pm \frac{1}{\Delta} \right) \log \frac{t}{2} + O(1),$$

where  $\mathfrak{a}$  is defined in (3).

The proof can be obtained as in [1]. It uses Stirling's formula and the properties of  $F_{\pm}$  from Proposition 1 after splitting the integral at  $t - 4\sqrt{t}$  and  $t + 4\sqrt{t}$ .

We make also use of the following result of Maier and Montgomery in [4] concerning moments of Dirichlet polynomials:

**Proposition 4.** *Consider a Dirichlet polynomial  $P(s) = \sum_{p \leq N} a(p)p^{-s}$ . For  $T \geq 3$  and  $\alpha \in \mathbb{R}$  let  $s_1, \dots, s_R \in \mathbb{C}$  with  $1 \leq |\Im(s_i - s_j)| \leq T$  for  $i \neq j$ , and  $\Re s_i \geq \alpha$  for  $1 \leq i \leq R$ .*

*Then, for every positive integer  $k$  with  $N^k \leq T$ , it holds that*

$$\sum_{r=1}^R |P(s_r)|^{2k} \ll T(\log T)^2 k! \left( \sum_{p \leq N} |a(p)|^2 p^{-2\alpha} \right)^k.$$

Our result relies further on the estimate in the following proposition.

**Proposition 5.** *Let  $T \geq e^{e^{33}}$ ,  $(\log \log T)^2 \leq V \leq \frac{\log T}{\log \log T}$ ,  $\eta = \frac{1}{\log V}$  and  $k = \left\lfloor \frac{2V}{3(1+\eta)} \right\rfloor$ .*

*Then we have*

$$k(\log(k \log \log T) - 2 \log(\eta V)) \leq -\frac{2}{3} V \log \frac{V}{\log \log T} + \frac{4}{3} V \log \log V + \frac{2}{3} V.$$

The proof is completely analogous to the elementary proof in [5], there Proposition 14 on page 11 and 12.

Now using Proposition 2, we can give an upper and lower bound for the number of zeros in a certain region around ordinate  $t$ .

**Proposition 6.** *(GRH) Let  $t \geq 25$ ,  $\Delta \geq 2$ ,  $0 < h \leq \sqrt{t}$  and  $\chi$  be a primitive character mod  $q$ . Then*

$$-\frac{\log(qt)}{2\pi\Delta} - \frac{1}{\pi} \Re \sum_{p \leq e^{2\pi\Delta}} \frac{\chi(p) \log(p)}{p^{\frac{1}{2}+it}} \hat{F}_- \left( \frac{\log p}{2\pi} \right) + O(\log \Delta) \leq N(t+h, \chi) - N(t-h, \chi) - \frac{h}{\pi} \log \frac{qt}{2\pi}$$

and

$$N(t+h, \chi) - N(t-h, \chi) - \frac{h}{\pi} \log \frac{qt}{2\pi} \leq \frac{\log(qt)}{2\pi\Delta} - \frac{1}{\pi} \Re \sum_{p \leq e^{2\pi\Delta}} \frac{\chi(p) \log(p)}{p^{\frac{1}{2}+it}} \hat{F}_+ \left( \frac{\log p}{2\pi} \right) + O(\log \Delta).$$

**Proof:**

We only show the upper bound, the lower bound estimate can be done in a complete analogous way.

We use the functions of Proposition 1 and the results from Propositions 2 and 3, we see analogously to [5] (there Proposition 15 from page 12 on):

$$N(t+h, \chi) - N(t-h, \chi) \leq \left(2h + \frac{1}{\Delta}\right) \frac{1}{2\pi} \log \frac{qt}{2\pi} + O(1) - \frac{1}{\pi} \Re \sum_{n \leq e^{2\pi\Delta}} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \hat{F}_+ \left(\frac{\log n}{2\pi}\right).$$

Here

$$\begin{aligned} \frac{1}{\pi} \Re \sum_{n \leq e^{2\pi\Delta}} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \hat{F}_+ \left(\frac{\log n}{2\pi}\right) &= \frac{1}{\pi} \Re \sum_{p \leq e^{2\pi\Delta}} \frac{\log p \chi(p)}{p^{\frac{1}{2}+it}} \hat{F}_+ \left(\frac{\log p}{2\pi}\right) \\ &\quad + \frac{1}{\pi} \Re \sum_{p \leq e^{2\pi\Delta}} \frac{\log p \chi(p)^2}{p^{1+2it}} \hat{F}_+ \left(\frac{\log p}{\pi}\right) + O(1) \\ &= \frac{1}{\pi} \Re \sum_{p \leq e^{2\pi\Delta}} \frac{\log p \chi(p)}{p^{\frac{1}{2}+it}} \hat{F}_+ \left(\frac{\log p}{2\pi}\right) + O(\log \Delta), \end{aligned}$$

and this finishes the proof.  $\square$

### 3 V-typical ordinates

The method of Soundararajan in [7] relies on the notion of  $V$ -typical ordinates. We modify this definition for our purposes and define  $V_{(\delta, \chi, q)}$ -typical ordinates as follows.

**Definition 1.** ( $V_{(\delta, \chi, q)}$ -typical).

Let  $q \in \mathbb{N}$  and  $\chi$  a character mod  $q$ . If  $\chi$  is nonprincipal, let it be induced by  $\chi_1$  mod  $q_1$ , let  $T > e$  and  $0 < \delta \leq 1$ .

Let  $V \in \left[(\log \log T)^2, \frac{\log T}{\log \log T}\right]$ .

An ordinate  $t \in [T, 2T]$  is called  $V_{(\delta, \chi, q)}$ -typical of order  $T$ , if the following properties hold:

$$(i) \quad \forall \sigma \geq \frac{1}{2}: \left| \sum_{n \leq x} \frac{\chi_1(n)\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log(\frac{x}{n})}{\log x} \right| \leq 2V \text{ with } x = T^{\frac{1}{\sigma}},$$

$$(ii) \quad \forall t' \in (t-1, t+1):$$

$$N(t' + h, \chi) - N(t' - h, \chi) \leq (1 + \delta)V \text{ with } h = \frac{\delta \pi V}{\log(q_1 T)} \text{ and } [t' - h, t' + h] \subseteq [t-1, t+1],$$

$$(iii) \quad \forall t' \in (t-1, t+1):$$

$$N(t' + h, \chi) - N(t' - h, \chi) \leq V \text{ with } h = \frac{\pi V}{\log V \log(q_1 T)} \text{ and } [t' - h, t' + h] \subseteq [t-1, t+1].$$

If at least one of the three properties does not hold, we call  $t$  a  $V_{(\delta, \chi, q)}$ -untypical ordinate of order  $T$ .

In what follows, the meaning of  $\chi$ ,  $q$  and  $\delta$  is often clear from the context, then we will write simply  $V$ -typical instead of  $V_{(\delta, \chi, q)}$ -typical of order  $T$ .

## 4 $V$ such that all $t \in [T, 2T]$ are $V$ -typical

**Proposition 7.** *Let  $t$  be sufficiently large and let  $0 < h \leq \sqrt{t}$ , let  $\chi$  be a primitive character mod  $q$ . Then*

$$\left| N(t+h, \chi) - N(t-h, \chi) - \frac{h}{\pi} \log \frac{qt}{2\pi} \right| \leq \frac{\log(qt)}{2 \log \log(qt)} + \left( \frac{1}{2} + o(1) \right) \frac{\log(qt) \log \log \log(qt)}{(\log \log(qt))^2} \text{ for } t \rightarrow \infty.$$

**Proof:** As in [5], we estimate the sum of Proposition 6 as follows:

$$\left| \frac{1}{\pi} \Re \sum_{p \leq e^{2\pi\Delta}} \frac{\log p \chi(p)}{p^{\frac{1}{2}+it}} \hat{F}_+ \left( \frac{\log p}{2\pi} \right) \right| \ll \sum_{p \leq e^{2\pi\Delta}} \frac{1}{\sqrt{p}} \ll \frac{e^{\pi\Delta}}{\Delta}. \quad (4)$$

Now set  $\Delta = \frac{1}{\pi} \log \frac{\log(qt)}{\log \log(qt)}$ . By estimate (4), we obtain

$$\begin{aligned} & \left| N(t+h, \chi) - N(t-h, \chi) - \frac{h}{\pi} \log \frac{qt}{2\pi} \right| \\ & \leq \frac{\log(qt)}{2(\log \log(qt) - \log \log \log(qt))} + O\left( \frac{\frac{\log(qt)}{\log \log(qt)}}{\log \log(qt) - \log \log \log(qt)} \right) \\ & = \frac{\log(qt)}{2 \log \log(qt)} \sum_{k=0}^{\infty} \left( \frac{\log \log \log(qt)}{\log \log(qt)} \right)^k + O\left( \frac{\log(qt)}{(\log \log(qt))^2} \right) \\ & = \frac{\log(qt)}{2 \log \log(qt)} + \frac{\log(qt) \log \log \log(qt)}{2(\log \log(qt))^2} (1 + o(1)) \end{aligned}$$

with an  $o(1)$ -term not depending on  $q$ , more precise, it is  $O((\log \log \log t)^{-1})$ .  $\square$

**Proposition 8.** *Let  $\chi$  be a character mod  $q$ ,  $q_1$  be the conductor of  $\chi$  and  $0 < \delta \leq 1$ . Further let  $T$  be sufficiently large, at least  $T \geq \max\{q^2, e^{e^9}\}$ , and let  $V$  be such that*

$$\frac{3}{4} + \frac{\log \log \log T}{\log \log T} \leq V \frac{\log \log T}{\log T} \leq 1$$

*holds. Then all ordinates  $t \in [T, 2T]$  are  $V$ -typical of order  $T$ .*

As a consequence of this proposition, we conclude that  $V$ -typical ordinates exist.

**Proof:**

We have to verify properties (i), (ii) and (iii) from Definition 1.

Ad (i):

Let  $f(u) := \sum_{2 \leq n \leq u} \frac{\Lambda(n) \chi_1(n)}{\sqrt{n} \log n}$ ,  $u \geq 2$ . Then (see [5], page 16):

$$|f(u)| \leq \sum_{2 \leq n \leq u} \frac{\Lambda(n)}{\sqrt{n} \log n} \ll \frac{\sqrt{u}}{\log u},$$



and from this we obtain

$$\sum_{n \leq x} \frac{\Lambda(n) \chi_1(n)}{\sqrt{n} \log n} \log \frac{x}{n} = \int_1^x \frac{f(u)}{u} du \ll \frac{\sqrt{x}}{\log x}.$$

Since  $x = T^{\frac{1}{V}} \leq T^{\frac{4 \log \log T}{3 \log T}} \leq (\log T)^2$ , we have

$$\left| \sum_{n \leq x} \frac{\chi_1(n) \Lambda(n)}{n^{\sigma+it} \log n} \frac{\log(\frac{x}{n})}{\log x} \right| \ll \frac{\sqrt{x}}{(\log x)^2} \ll \frac{\log T}{(\log \log T)^2} = o(V).$$

Ad (ii):

Let  $t' \in [t-1, t+1]$  and  $h = \frac{\delta \pi V}{\log(q_1 T)}$ . Since  $h = \frac{\delta \pi V}{\log(q_1 T)} \leq \pi V \leq \log T < \sqrt{T}$ , we can apply Proposition 7 on the primitive character  $\chi_1 \bmod q_1$  that induces  $\chi$ . We obtain, using  $q^2 \leq T$ , that

$$\begin{aligned} N(t' + h, \chi) - N(t' - h, \chi) &\leq \frac{h}{\pi} \log \frac{q_1 t'}{2\pi} + \frac{\log(q_1 t')}{2 \log \log(q_1 t')} + \left(\frac{1}{2} + o(1)\right) \frac{\log(q_1 t') \log \log \log(q_1 t')}{(\log \log(q_1 t'))^2} \\ &\leq \frac{h}{\pi} \log \frac{q_1 T}{\pi} + \frac{\log(2q_1 T)}{2 \log \log T} + \left(\frac{1}{2} + o(1)\right) \frac{\log(2q_1 T) \log \log \log T}{(\log \log T)^2} \\ &\leq \delta V + \frac{\log T^{3/2}}{2 \log \log T} + \left(\frac{1}{2} + o(1)\right) \frac{\log T^{3/2} \log \log \log T}{(\log \log T)^2} \\ &= \delta V + \frac{3 \log T}{4 \log \log T} + \left(\frac{3}{4} + o(1)\right) \frac{\log T \log \log \log T}{(\log \log T)^2} \\ &\leq \delta V + \frac{3 \log T}{4 \log \log T} + \frac{\log T \log \log \log T}{(\log \log T)^2} \\ &\leq (1 + \delta)V. \end{aligned}$$

Ad (iii):

Let  $t' \in [t-1, t+1]$  and  $h = \frac{\pi V}{\log V \log(q_1 T)}$ , then

$$\begin{aligned} N(t + h, \chi) - N(t - h, \chi) &\leq \frac{h}{\pi} \log \frac{q_1 t'}{2\pi} + \frac{\log(q_1 t')}{2 \log \log(q_1 t')} + \left(\frac{1}{2} + o(1)\right) \frac{\log(q_1 t') \log \log \log(q_1 t')}{(\log \log(q_1 t'))^2} \text{ by Prop. 7} \\ &\leq \frac{V}{\log V} + \frac{3 \log T}{4 \log \log T} + \left(\frac{3}{4} + o(1)\right) \frac{\log T \log \log \log T}{(\log \log T)^2} \text{ analogously to (ii)} \\ &= \frac{3 \log T}{4 \log \log T} + \left(\frac{3}{4} + o(1)\right) \frac{\log T \log \log \log T}{(\log \log T)^2} \\ &\leq \frac{3 \log T}{4 \log \log T} + \frac{\log T \log \log \log T}{(\log \log T)^2} \leq V. \end{aligned}$$

□

## 5 The number of $V$ -untypical, well separated ordinates

**Proposition 9.** *Let  $\chi \neq \chi_0$  be a character mod  $q$  and  $q_1$  be the conductor of  $\chi$ . Further let*

1.  *$T$  be large, at least  $T \geq q^2$ ,*
2.  *$0 < h \leq \sqrt{T}$ ,*
3.  *$(\log \log T)^2 \leq V \leq \frac{\log T}{\log \log T}$ ,*
4.  *$T \leq t_1 < t_2 < \dots < t_R \leq 2T$  and  $t_{r+1} - t_r \geq 1$  for  $1 \leq r < R$ ,*
5.  *$N(t_r + h, \chi) - N(t_r - h, \chi) - \frac{h}{\pi} \log \frac{q_1 t_r}{2\pi} \geq V + O(1)$  for  $1 \leq r \leq R$ .*

*Then*

$$R \ll T \exp \left( -\frac{2}{3} V \log \frac{V}{\log \log T} + \frac{4}{3} V \log \log V + O(V) \right).$$

**Proof:**

If  $q_1 = q$ , then  $\chi$  is primitive. If  $q_1 < q$ , then  $\chi$  is induced by a primitive character  $\chi_1$  mod  $q_1$ , and we have

$$N(t, \chi) = N(t, \chi_1).$$

Therefore we can apply the results from Proposition 6 for  $\chi_1$  and  $q_1$ . By the estimate from Proposition 6 we obtain

$$\begin{aligned} V + O(1) &\leq N(t_r + h, \chi_1) - N(t_r - h, \chi_1) - \frac{h}{\pi} \log \frac{q_1 t_r}{2\pi} \\ &\leq \frac{\log(2qT)}{2\pi\Delta} + \left| \frac{1}{\pi} \sum_{p \leq e^{2\pi\Delta}} \frac{\chi(p) \log p}{p^{\frac{1}{2} + it_r}} \hat{F}_+ \left( \frac{\log p}{2\pi} \right) \right| + O(\log \Delta), \quad \Delta \geq 2. \end{aligned}$$

If we define  $a(p) := \frac{\chi(p) \log p}{\pi} \hat{F}_+ \left( \frac{\log p}{2\pi} \right)$ , we have:

$$\left| \sum_{p \leq e^{2\pi\Delta}} \frac{a(p)}{p^{\frac{1}{2} + it_r}} \right| \geq V - \frac{\log(2qT)}{2\pi\Delta} + O(\log \Delta) + O(1),$$

where  $|a(p)| \leq 4$  holds by Proposition 1.

Let

$$\eta = \frac{1}{\log V} \text{ and } \Delta = \frac{(1 + \eta) \log(qT)}{2\pi V}.$$

Then we have

$$\exp(2\pi\Delta) = (qT)^{\frac{1+\eta}{V}} \leq T^{\frac{3(1+\eta)}{2V}} \text{ since } q \leq \sqrt{T},$$

hence

$$\log \Delta \ll \log \log T \leq \sqrt{V}.$$

We obtain

$$\begin{aligned} V - \frac{\log(2qT)}{2\pi\Delta} + O(\log \Delta) + O(1) &= V - \frac{V \log(2qT)}{(1+\eta) \log(qT)} + O(\sqrt{V}) \\ &\geq \frac{\eta V}{1+\eta} - \frac{\log 2}{(1+\eta) \log \log T} + O(\sqrt{V}) \geq \frac{1}{2}\eta V. \end{aligned}$$

So we have

$$\left| \sum_{p \leq e^{2\pi\Delta}} \frac{a(p)}{p^{\frac{1}{2}+it_r}} \right| \geq \frac{1}{2}\eta V \text{ for } 1 \leq r \leq R.$$

Let  $k \in \mathbb{N}$  with  $k \leq \left\lfloor \frac{2V}{3(1+\eta)} \right\rfloor$ . Then we can apply Proposition 4 with  $N = (qT)^{(1+\eta)/V}$  since  $(qT)^{k \frac{1+\eta}{V}} \leq T^{k \frac{3(1+\eta)}{2V}} \leq T$  for  $q^2 \leq T$ .

Raising to the  $2k$ -th power and summing over all  $r = 1, \dots, R$ , applying Proposition 4 for  $\alpha = \frac{1}{2}$  and  $N = \left\lfloor (qT)^{\frac{1+\eta}{V}} \right\rfloor$ , we obtain analogously to [5] (page 15):

$$R \left( \frac{\eta V}{2} \right)^{2k} \leq \sum_{r=1}^R \left| \sum_{p \leq (qT)^{\frac{1+\eta}{V}}} \frac{a(p)}{p^{\frac{1}{2}+it_r}} \right|^{2k} \ll T(\log T)^2 (Ck \log \log T)^k$$

with an absolute constant  $C > 0$ . So we have by now

$$R \ll T(\log T)^2 (4C)^k \left( \frac{k \log \log T}{\eta^2 V^2} \right)^k.$$

Now set  $k = \left\lfloor \frac{2V}{3(1+\eta)} \right\rfloor$ , and we obtain by Proposition 5:

$$\left( \frac{k \log \log T}{\eta^2 V^2} \right)^k \leq \exp \left( -\frac{2}{3} V \log \frac{V}{\log \log T} + \frac{4}{3} V \log \log V + \frac{2}{3} V \right).$$

With

$$(\log T)^2 (4C)^k = \exp(O(V)), \text{ see [5],}$$

we get the assertion with an absolute  $O$ -constant. □

**Proposition 10.** (GRH) Let  $\chi$  be a character mod  $q$  with conductor  $q_1$ . Further let  $T$  be large, let

$$2(\log \log T)^2 \leq V \leq \frac{\log T}{\log \log T},$$

and let  $T \leq t_1 < t_2 < \dots < t_R \leq 2T$  be  $V$ -untypical ordinates with  $t_{r+1} - t_r \geq 1$  for all  $1 \leq r < R$ . Then

$$R \ll T \exp \left( -\frac{2}{3} V \log \frac{V}{\log \log T} + \frac{4}{3} V \log \log V + O(V) \right)$$

with an  $O$ -constant independent of  $q$  and  $\chi$ .

**Proof:**

If  $t$  is a  $V$ -untypical ordinate, then at least one of the criteria of Definition 1 is false. For each criterion that is hurt, we give estimates for the corresponding number  $R_1$ ,  $R_2$  and  $R_3$  of such well-separated ordinates being counted in the Proposition.

If criterion (i) is false for  $t_r$ , then there exists a  $\sigma_r \geq \frac{1}{2}$  such that

$$\left| \sum_{n \leq x} \frac{\Lambda(n) \chi_1(n)}{n^{\sigma_r + it_r}} \frac{\log \frac{x}{n}}{\log n \log x} \right| > 2V,$$

note here that  $x = T^{\frac{1}{V}}$ .

The size of the sum over  $n = p^\alpha$  with  $\alpha \geq 2$  is

$$\begin{aligned} \left| \sum_{\substack{n=p^\alpha \leq x \\ \alpha \geq 2}} \frac{\Lambda(n) \chi_1(n)}{n^{\sigma_r + it_r}} \frac{\log \frac{x}{n}}{\log n \log x} \right| &\leq \sum_{p \leq \sqrt{x}} \frac{1}{p} + \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 3}} \frac{1}{p^{\frac{\alpha}{2}}} \\ &\ll \log \log x \ll \log \log T \ll \sqrt{V}. \end{aligned}$$

So if we count the ordinates  $t_r$  with

$$\left| \sum_{p \leq x} \frac{\chi_1(p)}{p^{\sigma_r + it_r}} \frac{\log \frac{x}{p}}{\log x} \right| \geq V,$$

where again  $x = T^{\frac{1}{V}}$ , we get an upper bound for  $R_1$ .

Now we apply Proposition 4 of Maier and Montgomery, we obtain

$$R_1 V^{2k} \leq \sum_{r \leq R} \left| \sum_{p \leq x} \frac{\chi_1(p)}{p^{\sigma_r + it_r}} \frac{\log \frac{x}{p}}{\log x} \right|^{2k} \ll T (\log T)^{2k} \left( \sum_{p \leq x} \frac{\log^2 \frac{x}{p}}{p \log^2 x} \right)^k,$$

where  $x^k \leq T$  holds for every  $k \leq V$ .

Now

$$\sum_{p \leq x} \frac{\log^2 \frac{x}{p}}{p \log^2 x} \leq \sum_{p \leq x} \frac{1}{p} \ll \log \log x \leq \log \log T.$$

As in [5], we obtain with  $k = \lfloor V \rfloor$ :

$$R_1 \ll T (\log T)^2 \left( \frac{Ck \log \log T}{V^2} \right)^k = T \exp \left( -V \log \frac{V}{\log \log T} + O(V) \right).$$

Now let (ii) be false, i. e. for  $t_r$  there exists a  $t'_r$  with  $|t_r - t'_r| \leq 1$  and

$$N \left( t'_r + \frac{\pi \delta V}{\log(q_1 T)}, \chi \right) - N \left( t'_r - \frac{\pi \delta V}{\log(q_1 T)}, \chi \right) > (1 + \delta) V.$$

With

$$\delta V = \frac{\delta V}{\log(q_1 T)} \log \left( \frac{q_1 t'_r}{2\pi} \right) + o(1) \text{ for } T \rightarrow \infty$$

we obtain

$$N \left( t'_r + \frac{\pi \delta V}{\log(q_1 T)}, \chi \right) - N \left( t'_r - \frac{\pi \delta V}{\log(q_1 T)}, \chi \right) - \frac{\delta V}{\log(q_1 T)} \log \left( \frac{q_1 t'_r}{2\pi} \right) \geq V + O(1).$$

Now we can apply Proposition 9, if the  $t'_r$  have a sufficiently large distance from another. So instead of the sequence  $t'_r$  being induced from  $t_r$  for  $1 \leq r \leq R_2$ , consider the three subsequences  $t'_{3s+\ell}$  with  $\ell \in \{1, 2, 3\}$ ,

$0 \leq s \leq \lfloor \frac{R_2 - \ell}{3} \rfloor$ , they have the property  $t'_{3(s+1)+\ell} - t'_{3s+\ell} \geq 1$ . We can apply Proposition 9 on any of the three subsequences and obtain

$$R_2 \leq 3 \left( \left\lfloor \frac{R_2}{3} \right\rfloor + 1 \right) + 2 \ll T \exp \left( -\frac{2}{3} V \log \left( \frac{V}{\log \log T} \right) + \frac{4}{3} V \log \log V + O(V) \right).$$

For  $R_3$  we obtain, analogously as in [5], the same bound with a similar calculation.  $\square$

## 6 Logarithmic derivative of $L(s, \chi)$

In this section, we consider only primitive characters.

**Proposition 11.** *Let  $\chi$  be a primitive character mod  $q$ ,  $T$  be sufficiently large,  $\frac{1}{2} \leq \sigma \leq 2$ ,  $T \leq t \leq 2T$  and  $L(\sigma + it, \chi) \neq 0$ . Then*

$$\Re \frac{L'}{L}(\sigma + it, \chi) = F(\sigma + it, \chi) - \frac{1}{2} \log(qT) + O(1),$$

where  $F(s, \chi) := \sum_{\rho} \Re \frac{1}{s - \rho}$  and the sum runs through all nontrivial zeros of  $L(s, \chi)$ .

**Proof:**

We use the formula

$$\frac{L'}{L}(s, \chi) = -\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \Re \frac{\Gamma'}{\Gamma} \left( \frac{s + \mathfrak{a}}{2} \right) + B(\chi) + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right)$$

that holds for primitive characters, where  $\Re B(\chi) = -\sum_{\rho} \Re(\frac{1}{\rho})$  and the sum runs through all nontrivial zeros  $\rho$  of  $L(s, \chi)$ . By Stirling's formula we obtain

$$\begin{aligned} \Re \frac{L'}{L}(\sigma + it, \chi) &= -\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \Re \frac{\Gamma'}{\Gamma} \left( \frac{\sigma + it + \mathfrak{a}}{2} \right) + \Re B(\chi) + \sum_{\rho} \Re \left( \frac{1}{\sigma + it - \rho} + \frac{1}{\rho} \right) \\ &= -\frac{1}{2} \log q - \frac{1}{2} \log |\sigma + it + \mathfrak{a}| + F(\sigma + it, \chi) + O(|\sigma + it + \mathfrak{a}|^{-1}) + O(1) \\ &= F(\sigma + it, \chi) - \frac{1}{2} \log(qT) + O(1). \end{aligned}$$

$\square$

**Proposition 12.** *Let  $\chi$  be a primitive character mod  $q$ . Let  $x \geq 1$ , and consider  $z \in \mathbb{C}$  that is not a pole of  $\frac{L'}{L}(z, \chi)$ . Then*

$$\sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n^z} \log \left( \frac{x}{n} \right) = -\frac{L'}{L}(z, \chi) \log x - \left( \frac{L'}{L} \right)'(z, \chi) - \sum_{\rho} \frac{x^{\rho-z}}{(\rho-z)^2} - \sum_{n \geq 0} \frac{x^{-2n-\mathfrak{a}-z}}{(z+2n+\mathfrak{a})^2}.$$

**Proof:** Since

$$\frac{L'}{L}(s, \chi) \ll \log(q|s|) \text{ for } \Re s \leq -\frac{1}{2} \text{ and } |s+m| > \frac{1}{4} \text{ for all } m \in \mathbb{N},$$

the proof works analogously to [5], where the term coming from the pole at  $s = 1$  is removed and the sum over the trivial zeros has been adjusted.  $\square$

Estimating the last sum analogously to [5], we obtain:

**Proposition 13.** *Let  $\chi$  be a primitive character mod  $q$ ,  $T \geq 1$  and  $1 \leq x \leq T$ . Let  $z \in \mathbb{C}$ ,  $\Re z \geq 0$ ,  $T \leq \Im z \leq 2T$ , and let  $z$  be not a pole of  $\frac{L'}{L}(z, \chi)$ .*

Then

$$\sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n^z} \log \left( \frac{x}{n} \right) = -\frac{L'}{L}(z, \chi) \log x - \left( \frac{L'}{L} \right)'(z, \chi) - \sum_{\rho} \frac{x^{\rho-z}}{(\rho-z)^2} + O(T^{-1}). \quad (5)$$

$\square$

## 7 Lower bound for $\log |L(s, \chi)|$

With the aid of  $V$ -typical ordinates, we estimate  $\log L(s, \chi)$  from below.

**Proposition 14.** (GRH) *Let  $\chi$  be a nonprincipal character mod  $q$  induced by  $\chi_1$  mod  $q_1$ . Let  $T$  be sufficiently large and  $T \leq t \leq 2T$ .*

Then for all  $\frac{1}{2} \leq \sigma \leq 2$  and  $2 \leq x \leq T$  it holds that

$$\log |L(\sigma + it, \chi)| \geq \Re \left( \sum_{n \leq x} \frac{\Lambda(n) \chi_1(n)}{n^{\sigma+it} \log n} \frac{\log \frac{x}{n}}{\log x} \right) - \left( 1 + \frac{x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log x} \right) \frac{F(\sigma + it, \chi)}{\log x} + O \left( \sqrt{\frac{\log q}{\log \log q}} \right),$$

where  $F$  is the function from Proposition 11.

**Proof:** At first, let  $\chi$  be primitive. By integrating equation (5) from  $z = \sigma + it$  to  $z = 2 + it$ , we obtain analogously to [5]:

$$\log |L(\sigma + it, \chi)| \geq \Re \left( \sum_{n \leq x} \frac{\Lambda(n) \chi(n)}{n^{\sigma+it} \log n} \frac{\log \frac{x}{n}}{\log x} \right) - \left( 1 + \frac{x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log x} \right) \frac{F(\sigma + it, \chi)}{\log x} + O(1).$$

Now let  $\chi$  mod  $q$  be not primitive and induced by the primitive character  $\chi_1$  mod  $q_1$ .

Then we have

$$L(s, \chi) = L(s, \chi_1) \prod_{p|q} \left( 1 - \frac{\chi_1(p)}{p^s} \right). \quad (6)$$

We obtain with equation (6):

$$\begin{aligned} \log |L(s, \chi)| &= \log |L(s, \chi_1)| + \sum_{p|q} \log \left| 1 - \frac{\chi_1(p)}{p^s} \right| \\ &\geq \Re \left( \sum_{n \leq x} \frac{\Lambda(n) \chi_1(n)}{n^{\sigma+it} \log n} \frac{\log \frac{x}{n}}{\log x} \right) - \left( 1 + \frac{x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log x} \right) \frac{F(\sigma + it, \chi_1)}{\log x} + O(1) + \sum_{p|q} \log \left| 1 - \frac{\chi(p)}{p^s} \right|. \end{aligned}$$

For the last sum we get

$$\sum_{p|q} \log \left| 1 - \frac{\chi(p)}{p^s} \right| \leq \sum_{p|q} \frac{1}{p^{1/2}} \leq \sum_{j=1}^{2 \log q} \frac{1}{p_j^{1/2}} \ll \sqrt{\frac{\log q}{\log \log q}}. \quad (7)$$

From equation (6) we see further that

$$F(s, \chi) = F(s, \chi_1),$$

so we get the stated bound.  $\square$

Now we would like to give an estimate for  $L(s, \chi)$  in the interval  $\Re(s) \in (\frac{1}{2}, 2)$ . For this, we split the interval at  $\frac{1}{2} + \frac{V}{\log T}$  and give a bound for each part. This is done in the next two propositions.

**Proposition 15.** (GRH) *Let  $\chi$  be a nonprincipal character mod  $q$ , and further let  $T$  be sufficiently large, at least  $T \geq q$ , let  $V \in [(\log \log T)^2, \frac{\log T}{\log \log T}]$  and let  $t \in [T, 2T]$  be  $V_{\delta, \chi, q}$ -typical of order  $T$ .*

*Then it holds for  $\frac{1}{2} + \frac{V}{\log T} \leq \sigma \leq 2$ , that*

$$\log |L(\sigma + it, \chi)| \geq f_{\delta, q}(V, \sigma + it),$$

where  $f_{\delta, q} : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}$ ,  $f_{\delta, q}(V, \sigma + it) = O\left(\frac{V}{\delta} + \sqrt{\frac{\log q}{\log \log q}}\right)$ .

**Proof:**

In Proposition 14 we set  $x = T^{\frac{1}{2}}$ . Then  $2 \leq x \leq T$ , and since  $\frac{1}{2} + \frac{V}{\log T} \leq \sigma$ , we have

$$\frac{x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log x} \leq \frac{\exp(-V \frac{\log x}{\log T})}{V \frac{\log x}{\log T}} = e^{-1} \leq 1.$$

Applying now Proposition 14, we obtain:

$$\log |L(\sigma + it, \chi)| \geq -2V - 2 \frac{V}{\log T} F(\sigma + it, \chi) + O\left(\sqrt{\frac{\log q}{\log \log q}}\right) =: f_{\delta, q}(V, \sigma + it),$$

since  $t$  is  $V$ -typical.

We aim to majorize  $F(\sigma + it, \chi)$  independent from  $q$  and  $\chi$ . As in [5], we divide the region of the zero-ordinates in two parts as follows.

- (i)  $\gamma$  with  $\frac{2\pi n \delta V}{\log(q_1 T)} \leq |t - \gamma| \leq \frac{2\pi(n+1)\delta V}{\log(q_1 T)}$  for  $0 \leq n \leq N = \left\lfloor \frac{\log(q_1 T)}{4\pi \delta V} \right\rfloor$ ,
- (ii)  $\gamma$  with  $\{\gamma : |\gamma - t| \geq \frac{1}{2}\}$ , where  $q_1$  denotes the conductor of  $\chi \bmod q$ .

Consider the set of  $\gamma$  from (i):

$$\begin{aligned} \sum_{\gamma \text{ from (i)}} \Re \frac{1}{\sigma + it - \frac{1}{2} - i\gamma} &= 2 \sum_{\gamma \text{ from (i)}} \frac{(\sigma - \frac{1}{2})}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \\ &\leq 2(1 + \delta)V \sum_{n=0}^N \frac{(\sigma - \frac{1}{2})}{(\sigma - \frac{1}{2})^2 + \left(\frac{2\pi n \delta V}{\log(q_1 T)}\right)^2} \text{ since } t \text{ is } V\text{-typical, (ii),} \\ &\leq 4V \left( \frac{1}{\sigma - \frac{1}{2}} + \frac{\log(q_1 T)}{4\delta V} \right), \end{aligned}$$

since for  $a, c \in \mathbb{R}_{>0}$  and  $N \in \mathbb{N}$  we have  $\sum_{n=0}^N \frac{a}{a^2 + (cn)^2} \leq \frac{1}{a} + \frac{\pi}{2c}$ , see [5] Prop. 6, and we continue with

$$\leq 4 \log(q_1 T) + \frac{\log(q_1 T)}{\delta} \leq 5 \frac{\log(qT)}{\delta}.$$

For the sum over  $\gamma$  with (ii) we work with the known formula

$$\sum_{\rho \in \mathcal{N}(\chi)} \frac{1}{1 + (t - \Im(\rho))^2} \ll \log(q(2 + |t|)) \quad (8)$$

holding for primitive characters mod  $q$ . Since  $\mathcal{N}(\chi) = \mathcal{N}(\chi_1)$  if  $\chi \bmod q$  is induced by  $\chi_1 \bmod q_1 \leq q$ , we can use this formula also in the case of a nonprimitive character mod  $q$ .

For  $0 \leq \sigma - \frac{1}{2} \leq \frac{3}{2}$  and  $|t - \gamma| \geq \frac{1}{2}$  we have

$$\frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \leq \frac{8}{1 + (t - \gamma)^2}, \quad (9)$$

therefore we can estimate the sum over  $\gamma$  with (ii) using (9) and (8) as follows:

$$\begin{aligned} \sum_{|t - \gamma| \geq \frac{1}{2}} \Re\left(\frac{1}{\sigma + it - \frac{1}{2} - i\gamma}\right) &= \sum_{|t - \gamma| \geq \frac{1}{2}} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \\ &\leq \sum_{|t - \gamma| \geq \frac{1}{2}} \frac{8}{1 + (t - \gamma)^2} \leq \sum_{\rho \in \mathcal{N}(\chi)} \frac{8}{1 + (t - \Im(\rho))^2} \ll \log(qt). \end{aligned}$$

Now consider  $g(x) := \frac{\log(qx)}{\log x}$ , we see that  $g(x)$  is monotonously decreasing for  $x > 1$ , and so for  $x \geq q$  we have  $g(x) \leq g(q) = 2$ .

We resume the two results for the regions (i) and (ii) as follows:

$$\left| 2 \frac{V}{\log T} F(\sigma + it, \chi) \right| \ll \frac{\log(qT) V}{\log T} \frac{1}{\delta} \ll \frac{V}{\delta} \text{ since } q \leq T,$$

which gives the asserted bound for  $f_{\delta, q}(V, \sigma + it)$ . □

**Proposition 16.** (GRH) Let  $\chi$  be a character mod  $q$ , let  $T$  be sufficiently large,  $V \in [(\log \log T)^2, \frac{\log T}{\log \log T}]$  and  $t \in [T, 2T]$  be  $V$ -typical (of order  $T$ ).

Then we have for all  $\frac{1}{2} < \sigma \leq \sigma_0 = \frac{1}{2} + \frac{V}{\log T}$ :

$$\log |L(\sigma + it, \chi)| \geq \log |L(\sigma_0 + it, \chi)| - V \log \frac{\sigma_0 - \frac{1}{2}}{\sigma - \frac{1}{2}} - 2(1 + \delta)V \log \log V + O\left(\frac{V}{\delta^2} + \sqrt{\frac{\log q}{\log \log q}}\right).$$

**Proof:**

Consider at first a primitive character  $\chi \bmod q$ , i. e.  $q_1 = q$ . We work as in [5], p. 8, and get:

$$\log |L(\sigma_0 + it, \chi)| - \log |L(\sigma + it, \chi)| \leq \frac{1}{2} \sum_{\gamma} \log \frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2}.$$

In order to estimate the sum, we divide the set of  $\gamma$  in three subsets such that we can make use of the fact that  $t$  is a  $V$ -typical ordinate.

The division of the  $\gamma$  is as follows.

(a)  $\gamma$  with  $|t - \gamma| \leq \frac{\pi V}{\log V \log(qT)},$



- (b)  $\gamma$  with  $(2\pi\delta n + \frac{\pi}{\log V}) \frac{V}{\log(qT)} \leq |t - \gamma| \leq (2\pi\delta(n+1) + \frac{\pi}{\log V}) \frac{V}{\log(qT)}$  ( $0 \leq n \leq N = \lfloor \frac{\log(qT)}{4\pi\delta V} \rfloor$ ),
- (c)  $\gamma$  with  $\{\gamma : |t - \gamma| \geq \frac{1}{2}\}$ .

Since  $\sigma \leq \sigma_0$ , we have

$$\frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \leq \frac{(\sigma_0 - \frac{1}{2})^2}{(\sigma - \frac{1}{2})^2}.$$

For the  $\gamma$  from (a) we use property (iii) from the definition of  $V$ -typical and obtain

$$\frac{1}{2} \sum_{|t-\gamma| \leq \frac{\pi V}{\log V \log(qT)}} \log \frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \leq \frac{1}{2} \sum_{|t-\gamma| \leq \frac{\pi V}{\log V \log(qT)}} \log \frac{(\sigma_0 - \frac{1}{2})^2}{(\sigma - \frac{1}{2})^2} \leq V \log \frac{\sigma_0 - \frac{1}{2}}{\sigma - \frac{1}{2}}.$$

We use the fact that  $\frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2}$  is decreasing in  $|t - \gamma|$ . With this, we estimate the set of  $\gamma$  in (b) using property (ii) in the definition of  $V$ -typical. For the  $\gamma$  with (c) we use the general zero estimate for  $L(s, \chi)$  and obtain in the same way as in [5]:

$$\frac{1}{2} \sum_{\gamma \text{'s in (b)}} \log \frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \leq 2(1 + \delta)V \log \log V + O\left(\frac{V}{\delta^2}\right)$$

and

$$\frac{1}{2} \sum_{|\gamma - t| \geq \frac{1}{2}} \log \frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \ll \frac{V}{\log \log T}.$$

This gives the assertion for primitive characters.

Now if  $\chi$  is not primitive mod  $q$  and induced by the primitive character  $\chi_1$  mod  $q_1$ , we use equation (7) and obtain

$$\begin{aligned} \log |L(\sigma + it, \chi)| &= \log |L(\sigma + it, \chi_1)| + O\left(\sqrt{\frac{\log q}{\log \log q}}\right) \\ &\geq \log |L(\sigma_0 + it, \chi_1)| - V \log \frac{\sigma_0 - \frac{1}{2}}{\sigma - \frac{1}{2}} - 2(1 + \delta)V \log \log V + O\left(\frac{V}{\delta^2}\right) + O\left(\sqrt{\frac{\log q}{\log \log q}}\right) \\ &= \log |L(\sigma_0 + it, \chi)| + -V \log \frac{\sigma_0 - \frac{1}{2}}{\sigma - \frac{1}{2}} - 2(1 + \delta)V \log \log V + O\left(\frac{V}{\delta^2}\right) + \sqrt{\frac{\log q}{\log \log q}}. \end{aligned}$$

□

At the end of this section we combine the results from propositions 8, 15 and 16. With these, we obtain a lower bound for the whole stripe  $\Re(s) \in (\frac{1}{2}, 2)$ .

**Proposition 17.** (GRH) Let  $\chi$  be a character mod  $q$ ,  $|t|$  be sufficiently large, at least  $|t| \geq q$ , and  $\frac{1}{2} < \sigma \leq 2$ . Then

$$\log |L(\sigma + it, \chi)| \geq -\frac{\log |t|}{\log \log |t|} \log \frac{1}{(\sigma - \frac{1}{2})} - 3 \frac{\log |t| \log \log \log |t|}{\log \log |t|}.$$

**Proof:**

As in [5], we choose

$$V = \frac{\log |t|}{\log \log |t|} \text{ and } \delta = \frac{1}{2},$$

note that then  $O\left(\frac{V}{\delta^2} + \sqrt{\frac{\log q}{\log \log q}}\right) = O(V)$ . □

By now, we gave estimates for  $L(s, \chi)$  in a region for sufficiently large  $\Im(s)$ . We also need an estimate for  $L(s, \chi)$  in the remaining region, which we give in the next Proposition.

**Proposition 18.** (GRH) *Let  $x$  be large,  $c > 0$ . Further let  $T_0(x) := T_0 := 2^{\lfloor (\log x)^{3/5} (\log \log x)^c \rfloor}$ , and  $\sigma = \frac{1}{2} + \frac{1}{\log x}$ . Then there exists a  $C > 0$ , such that for all  $|t| \leq T_0$ ,  $q \leq \sqrt{T_0}$  and a nonprincipal character  $\chi \pmod{q}$  we have*

$$|L(\sigma + it, \chi)| \geq T_0^{-C \log \log x}.$$

**Proof:**

At first, let  $\chi$  be a primitive character mod  $q$ , and  $q \leq \sqrt{T_0}$ . By the explicit formula for the logarithmic derivation of  $L$  we obtain

$$\int_{\sigma+it}^{2+it} \frac{L'}{L}(s+it, \chi) ds = \int_{\sigma+it}^{2+it} \left( \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |\Im(s) - \Im(\rho)| \leq 1}} \frac{1}{s - \rho} + O(\log(q(2 + |\Im(s)|))) \right) ds,$$

hence

$$\begin{aligned} & \log L(2+it, \chi) - \log L(\sigma+it, \chi) \\ &= \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leq 1}} \log(2+it - \rho) - \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leq 1}} \log(\sigma+it - \rho) + O(\log(q(2 + |t|))). \end{aligned}$$

Considering the real parts, it follows that

$$\log |L(\sigma+it, \chi)|^{-1} = \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leq 1}} \log \left| \frac{3}{2} + i(t - \Im(\rho)) \right| + \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leq 1}} \log \frac{1}{|\sigma+it - \rho|} + O(\log(q(2 + |t|))).$$

To give an estimate of the first sum, we have

$$\left| \frac{3}{2} + i(t - \Im(\rho)) \right| \leq \frac{5}{2} \text{ for } |t - \Im(\rho)| \leq 1, \text{ hence } \sum_{|t - \Im(\rho)| \leq 1} \log \left| \frac{3}{2} + i(t - \Im(\rho)) \right| \ll \log(qt),$$

and to give an estimate for the second sum, we have

$$\begin{aligned} |\sigma+it - \rho|^{-1} &= \left| \frac{1}{\log x} + i(t - \Im(\rho)) \right|^{-1} \leq \log x, \\ \text{hence } \sum_{|t - \Im(\rho)| \leq 1} \log \frac{1}{|\sigma+it - \rho|} &\ll \log(qt) \log \log x. \end{aligned}$$

Therefore we obtain

$$\log |L(\sigma+it, \chi)|^{-1} \ll \log(qt) \log \log x.$$

If we note that  $t \leq T_0$  and  $q \leq \sqrt{T_0}$ , we obtain

$$\log |L(\sigma + it, \chi)|^{-1} \ll \log T_0 \log \log x.$$

This gives the assertion for primitive characters.

Now let  $\chi$  be a nonprimitive character mod  $q$  and induced by  $\chi_1$  mod  $q_1$ . We conclude:

$$\begin{aligned} \log |L(\sigma + it, \chi)|^{-1} &= \log |L(\sigma + it, \chi_1)|^{-1} - \sum_{p|q} \log \left| 1 - \frac{\chi(p)}{p^s} \right| \\ &= \log |L(\sigma + it, \chi_1)|^{-1} + O\left(\sqrt{\frac{\log T_0}{\log \log T_0}}\right) \\ &\ll \log T_0 \log \log x \left(1 + O\left(\frac{1}{\sqrt{\log T_0 \log \log T_0 \log \log x}}\right)\right) \\ &\ll \log T_0 \log \log x. \end{aligned}$$

□

## 8 Majorant of $|x^z L(z, \chi)^{-1}|$

In this section we give a majorant of  $|x^z L(z, \chi)^{-1}|$  for certain  $z$ . It is a consequence of Propositions 15 and 16.

**Proposition 19.** (GRH) *Let  $\chi$  be a character mod  $q$ . Further let  $t$  be sufficiently large (at least  $t \geq q$ ),  $x \geq t$ ,  $V' \in \left[(\log \log t)^2, \frac{\log(t/2)}{\log \log(t/2)}\right]$ ,  $V \geq V'$ ,  $t$  be  $V'$ -typical of order  $T'$ .*

*Then for  $V' \leq (\Re z - \frac{1}{2}) \log x \leq V$ ,  $|\Im z| = t$ , we have*

$$\left| x^z L(z, \chi)^{-1} \right| \leq \sqrt{x} \exp \left( V \log \frac{\log x}{\log t} + 2(1 + \delta)V \log \log V + O\left(V \delta^{-2} + \sqrt{\frac{\log x}{\log \log x}}\right) \right).$$

*Proof:* By taking notion of the changed error term, everything remains as in [5], see Proposition 22 there. □

## 9 Upper bound for $M(x, q, a)$

We need some preliminaries for the proof of the theorem.

For a character  $\chi$  mod  $q$ , let

$$A(x, \chi, q) := \frac{1}{2\pi i} \int_{1 + \frac{1}{\log x} - i2^K}^{1 + \frac{1}{\log x} + i2^K} \frac{x^s}{L(s, \chi)s} ds, \text{ where } K := \left\lceil \frac{\log x}{\log 2} \right\rceil,$$

and by Perron's formula we have:

$$M(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi(q)} \bar{\chi}(a) A(x, \chi, q) + O(\log x). \quad (10)$$

We aim to give a good upper bound for  $A(x, \chi, q)$ .

Further we assume w.l.o.g., that  $x \geq q^2$ , as otherwise we can estimate trivially.

Now we give some definitions being valid during this section.

**Definition 2.**

$$K := \left\lfloor \frac{\log x}{\log 2} \right\rfloor, \kappa := \left\lfloor (\log x)^{3/5} (\log \log x)^c \right\rfloor, \\ T_k := 2^k \text{ for } \kappa \leq k \leq K, \text{ so } q^2 \leq T_\kappa \leq T_K.$$

For  $k$  with  $\kappa \leq k < K$  and for  $n \in \mathbb{N} \cap [T_k, 2T_k)$ , we define the integer  $V_n$  to be the smallest integer in the interval  $\left[ (\log \log T_k)^2 + 1, \frac{\log T_k}{\log \log T_k} \right]$ , such that all points in  $[n, n+1]$  are  $V_n$ -typical ordinates of order  $T_k$ . The existence of these  $V_n$  is obtained by Proposition 8.

**Lemma 1.** Let  $x \geq 2$ ,  $c > 1$ ,  $q \in \mathbb{N}$  and  $1 < q \leq 2^{\kappa/2}$ . Further let  $\chi$  be a nonprincipal character mod  $q$  and  $\delta \in (0, 1]$ . Then

$$\frac{A(x, \chi, q)}{\sqrt{x}} \ll_\delta \exp \left( (\log x)^{3/5} (\log \log x)^{c+1+\delta} \right) + B(x, \chi, q),$$

where

$$B(x, \chi, q) = \sum_{n=T_\kappa}^{T_K-1} \frac{1}{n} \exp \left( V_n \log \left( \frac{\log x}{\log n} \right) + 2(1+2\delta)V_n \log \log V_n + D \sqrt{\frac{\log x}{\log \log x}} \right)$$

with an absolute constant  $D > 0$ .

**Proof:**

We choose the following path of integration  $S(x, \chi, q)$ , we describe it for the upper half plane  $\Im(z) \geq 0$ , it passes out analogously in the lower half plane.

1. A vertical segment  $\left[ \frac{1}{2} + \frac{1}{\log x}, \frac{1}{2} + \frac{1}{\log x} + iT_\kappa \right]$ .
2. Further vertical segments  $\left[ \frac{1}{2} + \frac{V_n}{\log x} + in, \frac{1}{2} + \frac{V_n}{\log x} + i(n+1) \right]$ .
3. A horizontal segment  $\left[ \frac{1}{2} + \frac{1}{\log x} + iT_\kappa, \frac{1}{2} + \frac{V_{T_\kappa}}{\log x} + iT_\kappa \right]$ .
4. Additional horizontal segments for  $T_\kappa \leq n \leq T_K - 2$ , namely  $\left[ \frac{1}{2} + \frac{V_n}{\log x} + i(n+1), \frac{1}{2} + \frac{V_{n+1}}{\log x} + i(n+1) \right]$ .
5. The last horizontal segment  $\left[ \frac{1}{2} + \frac{V_{T_K-1}}{\log x} + iT_K, 1 + \frac{1}{\log x} + iT_K \right]$ .

Hence

$$|A(x, \chi, q)| = \frac{1}{2\pi} \left| \int_{S(x, \chi, q)} \frac{x^s}{L(s, \chi)s} ds \right|.$$

We consider just the first segment more accurate, the others can be estimated analogously to [5]:

Ad 1.:

$$\begin{aligned}
\frac{1}{2\pi} \left| \int_{\substack{S(x, \chi, q) \\ |\Im(z)| \leq T_\kappa}} \frac{x^s}{L(s, \chi)s} ds \right| &\leq \frac{1}{2\pi} x^{\frac{1}{2} + \frac{1}{\log x}} \int_{-T_\kappa}^{T_\kappa} \left| L\left(\frac{1}{2} + \frac{1}{\log x} + it, \chi\right) \right|^{-1} \frac{dt}{\sqrt{\frac{1}{4} + t^2}} \\
&\leq \frac{e}{2\pi} \sqrt{x} \max_{|t| \leq T_\kappa} \left| L\left(\frac{1}{2} + \frac{1}{\log x} + it, \chi\right) \right|^{-1} \int_{-T_\kappa}^{T_\kappa} \frac{dt}{\sqrt{\frac{1}{4} + t^2}} \\
&\leq \sqrt{x} \max_{|t| \leq T_\kappa} \left| L\left(\frac{1}{2} + \frac{1}{\log x} + it, \chi\right) \right|^{-1} \int_0^{T_\kappa} \frac{dt}{\sqrt{\frac{1}{4} + t^2}} \\
&\leq 2\sqrt{x} \max_{|t| \leq T_\kappa} \left| L\left(\frac{1}{2} + \frac{1}{\log x} + it, \chi\right) \right|^{-1} \log T_\kappa \\
&\ll \sqrt{x} (\log T_\kappa) T_\kappa^{C \log \log x} \text{ by Prop. 18} \\
&\leq \sqrt{x} T_\kappa^{C_1 \log \log x} \text{ with } C_1 = C + 1.
\end{aligned}$$

Ad 2.:

$$\begin{aligned}
\frac{1}{2\pi} \left| \int_{\substack{\frac{1}{2} + \frac{V_n}{\log x} + i(n+1) \\ \frac{1}{2} + \frac{V_n}{\log x} + in}} \frac{x^s}{L(s, \chi)s} ds \right| &\leq \frac{1}{2\pi n} \max_{\substack{z \in \{\frac{1}{2} + \frac{V_n}{\log x} + it; \\ t \in [n, n+1]\}} \left| x^z L(z, \chi)^{-1} \right| \text{ as } |s| \geq |n| \\
&\leq \frac{1}{n} \sqrt{x} \exp \left( V_n \log \left( \frac{\log x}{\log n} \right) + 2(1 + \delta) V_n \log \log V_n + D \left( \frac{V_n}{\delta^2} + \sqrt{\frac{\log x}{\log \log x}} \right) \right),
\end{aligned}$$

where  $D > 0$  is an absolute constant, see Proposition 19.

Ad 3.:

$$\frac{1}{2\pi} \left| \int_{\substack{\frac{1}{2} + \frac{V_{T_\kappa}}{\log x} + iT_\kappa \\ \frac{1}{2} + \frac{1}{\log x} + iT_\kappa}} \frac{x^s}{L(s, \chi)s} ds \right| \leq \sqrt{x} T_\kappa^3 \text{ by Prop. 17.}$$

Ad 4.:

Here we use Proposition 19 for  $n$  with  $T_\kappa \leq n \leq T_K - 2$ :

$$\begin{aligned}
&\left| \int_{\substack{\frac{1}{2} + \frac{V_{n+1}}{\log x} + i(n+1) \\ \frac{1}{2} + \frac{V_n}{\log x} + i(n+1)}} \frac{x^s}{L(s, \chi)s} ds \right| \\
&\leq \frac{1}{n} \sqrt{x} \exp \left( V_n \log \left( \frac{\log x}{\log n} \right) + 2(1 + \delta) V_n \log \log V_n + D \left( \frac{V_n}{\delta^2} + \sqrt{\frac{\log x}{\log \log x}} \right) \right) \\
&+ \frac{1}{n+1} \sqrt{x} \exp \left( V_{n+1} \log \left( \frac{\log x}{\log(n+1)} \right) + 2(1 + \delta) V_{n+1} \log \log V_{n+1} + D \left( \frac{V_{n+1}}{\delta^2} + \sqrt{\frac{\log x}{\log \log x}} \right) \right).
\end{aligned}$$

Ad 5.:

We obtain using Proposition 15:

$$\frac{1}{2\pi} \left| \int_{\frac{1}{2} + \frac{V_{T_K-1}}{\log x} + iT_K}^{1 + \frac{1}{\log x} + iT_K} \frac{x^s}{L(s, \chi)s} ds \right| \leq_\delta \sqrt{x}.$$

□

The following proposition is similar to Proposition 23 in [5], the modification here is necessary, but the proof works analogously.

**Proposition 20.** *Let  $A, C > 0$  and let  $A \geq 4C^4 + 1$ , then for  $V > e^{3C/2}$  it holds that*

$$AV - \frac{2}{3}V \log V + CV \log \log V \leq e^{3A/2} \left( \frac{3}{2}A \right)^{3C/2}.$$

**Lemma 2.** *Under the conditions of Lemma 1 we have*

$$B(x, \chi, q) \ll_\delta \exp \left( (\log x)^{3/5} (\log \log x)^{13/2 - 3c/2 + 8\delta} \right).$$

**Proof:**

We define for  $\kappa \leq k < K$ :

$$B(T_k, x, \chi, q) := \sum_{T_k \leq n < 2T_k} \frac{1}{n} \exp \left( V_n \log \left( \frac{\log x}{\log n} \right) + 2(1 + 2\delta)V_n \log \log V_n \right),$$

then

$$B(x, \chi, q) \leq K \max_{\kappa \leq k < K} B(T_k, x, \chi, q) \exp \left( D \sqrt{\frac{\log x}{\log \log x}} \right) \ll \log x \max_{\kappa \leq k < K} B(T_k, x, \chi, q) \exp \left( D \sqrt{\frac{\log x}{\log \log x}} \right),$$

so it remains to estimate  $B(T_k, x, \chi, q)$ .

To simplify the notation, we write now  $T_k = T$ ,  $a(T) := (\log \log T)^2$ ,  $b(T) := \frac{\log T}{\log \log T}$  and  $\mathcal{V}(V, T) := \{n \in \mathbb{N}; T \leq n < 2T, V_n = V\}$ .

We sort the summands corresponding to the values of the  $V_n$ :

$$\begin{aligned} B(T, x, \chi, q) &= \sum_{\substack{V \in \mathbb{N} \\ a(T) \leq V \leq b(T)}} \sum_{\substack{T \leq n < 2T \\ V_n = V}} \frac{1}{n} \exp \left( V \log \left( \frac{\log x}{\log n} \right) + 2(1 + 2\delta)V \log \log V \right) \\ &\leq \frac{1}{T} \sum_{\substack{V \in \mathbb{N} \\ a(T) \leq V \leq b(T)}} \exp \left( V \log \left( \frac{\log x}{\log T} \right) + 2(1 + 2\delta)V \log \log V \right) \text{card } \mathcal{V}(V, T). \end{aligned} \quad (11)$$

Now we split the sum over  $V$ . For  $V \leq 2a(T) + 1$  we use the trivial estimate

$$\text{card}\{n \in \mathbb{N}; T \leq n < 2T, V_n = V\} \leq T. \quad (12)$$

Then we estimate the corresponding sum for this part:

$$\frac{1}{T} \sum_{\substack{V \in \mathbb{N} \\ a(T) \leq V \leq 2a(T)+1}} \exp \left( V \log \left( \frac{\log x}{\log T} \right) + 2(1 + 2\delta)V \log \log V \right) \text{card } \mathcal{V}(V, T) = \exp \left( O((\log \log x)^3) \right). \quad (13)$$

Now consider  $V \in \mathbb{N}$  with  $2a(T) + 1 < V \leq b(T)$ , we split

$$\mathcal{V}(V, T) = \{n \equiv 0 \pmod{2}; n \in \mathcal{V}(V, T)\} \cup \{n \equiv 1 \pmod{2}; n \in \mathcal{V}(V, T)\} =: \mathcal{V}_0(V, T) \cup \mathcal{V}_1(V, T).$$

Consider a number  $n \in \mathcal{V}(V, T)$  for a fixed  $V$  with  $2a(T) + 1 < V \leq b(T)$ . Since  $V_n = V$  is the smallest integer such that all  $t \in [n, n+1]$  are  $V_n$ -typical of order  $T$ , there exists at least one  $t_n \in [n, n+1]$  being  $(V_n - 1)$ -untypical of order  $T$ .

So choose for any  $n \in \mathcal{V}(V, T)$  a  $t_n \in [n, n+1]$  being  $(V - 1)$ -untypical. This assignment gives a bijection between  $\mathcal{V}(V, T)$  and the set

$$\mathcal{U}(V, T) := \{t_n; n \in \mathcal{V}(V, T), t_n \in [n, n+1] \text{ and } t_n \text{ is } (V - 1)\text{-untypical}\}$$

of  $(V - 1)$ -untypical ordinates. Hence the cardinalities of both sets are equal, and in  $\mathcal{U}(V, T)$  all elements are  $(V - 1)$ -untypical of order  $T$ .

Further we define for  $h \in \{0, 1\}$  the set

$$\mathcal{U}_h(V, T) := \{t_n \in \mathcal{U}(V, T); n \in \mathcal{V}_h(V, T)\}.$$

For  $t_n \neq t_m$  with  $t_n, t_m \in \mathcal{U}_h(V, T)$  we have  $|t_n - t_m| \geq 1$ : If w.l.o.g.  $n < m$ , then  $t_m - t_n \geq m - (n+1) \geq 1$  since  $t_n \in [n, n+1]$ ,  $t_m \in [m, m+1]$  and  $n \equiv m \pmod{2}$ . So the sets  $\mathcal{U}_h(V, T)$  are sets of well distanced  $(V - 1)$ -untypical ordinates in the sense of Proposition 10.

Since  $\text{card } \mathcal{V}(V, T) = \text{card } \mathcal{U}(V, T) = \text{card } \mathcal{U}_0(V, T) + \text{card } \mathcal{U}_1(V, T)$ , we can estimate the cardinality measure of the set  $\mathcal{V}(U, T)$  using Proposition 10, we obtain

$$\begin{aligned} \text{card } \mathcal{V}(V, T) &\ll T \exp\left(-\frac{2}{3}(V-1) \log\left(\frac{V-1}{\log \log T}\right) + \frac{4}{3}(V-1) \log \log(V-1) + O(V)\right) \\ &\ll_\delta T \exp\left(-\frac{2}{3}V \log\left(\frac{V}{\log \log T}\right) + \left(\frac{4}{3} + \delta\right)V \log \log V\right). \end{aligned} \quad (14)$$

This leads to the following result:

$$\begin{aligned} B(T, x, \chi, q) &\leq \exp\left(O((\log \log x)^3)\right) \\ &+ \sum_{\substack{V \in \mathbb{N} \\ 2a(T)+1 \leq V \leq b(T)}} \frac{1}{T} \exp\left(V \log\left(\frac{\log x}{\log T}\right) + 2(1+2\delta)V \log \log V\right) \text{card } \mathcal{V}(V, T) \text{ by (11) and (13)} \\ &\ll_\delta \exp\left(O((\log \log x)^3)\right) \\ &+ \sum_{\substack{V \in \mathbb{N} \\ 2a(T)+1 \leq V \leq b(T)}} \exp\left(V \log\left(\frac{\log x (\log \log T)^{2/3}}{\log T}\right) - \frac{2}{3}V \log V + \left(\frac{10}{3} + 5\delta\right)V \log \log V\right) \end{aligned} \quad (15)$$

$$\begin{aligned} &\ll_\delta \exp\left(O((\log \log x)^3)\right) \\ &+ \sum_{\substack{V \in \mathbb{N} \\ 2a(T)+1 \leq V \leq b(T)}} \exp\left(V \log\left(\frac{\log x \log \log T}{\log T}\right) - \frac{2}{3}V \log V + \left(\frac{10}{3} + 5\delta\right)V \log \log V\right), \end{aligned} \quad (16)$$

where in (15) the implicit constant in the estimate depends on  $\delta$  since we used equation (14).

In order to majorize the last sum (16), we use Proposition 20 with the following parameters:

$$\text{Let } A := \log\left(\frac{\log x \log \log T}{\log T}\right) \text{ and } C := \frac{10}{3} + 5\delta.$$

(Then  $A \geq 4C^4 + 1$  and  $V > e^{3C/2}$  hold if  $x$  is large enough.)

We obtain

$$\begin{aligned} & \sum_{\substack{V \in \mathbb{N} \\ 2a(T)+1 \leq V \leq b(T)}} \exp \left( V \log \left( \frac{\log x \log \log T}{\log T} \right) - \frac{2}{3} V \log V + \left( \frac{10}{3} + 5\delta \right) V \log \log V \right) \\ & \leq \frac{\log T}{\log \log T} \exp \left( \left( \log x \frac{\log \log T}{\log T} \right)^{3/2} \left( \frac{3}{2} \log \left( \log x \frac{\log \log T}{\log T} \right) \right)^{5+15\delta/2} \right). \end{aligned} \quad (17)$$

Since

$$\frac{\log \log T}{\log T} = \frac{\log \log T_k}{\log T_k} \leq \frac{\log \log T_\kappa}{\log T_\kappa} \ll \frac{\log \log x}{(\log x)^{3/5} (\log \log x)^c} \leq (\log x)^{-3/5},$$

we have

$$\left( \log x \frac{\log \log T}{\log T} \right)^{3/2} \leq \left( (\log x)^{2/5} (\log \log x)^{1-c} \right)^{3/2} = (\log x)^{3/5} (\log \log x)^{3/2-3c/2},$$

and as  $c \geq 1$ , we obtain further

$$\left( \frac{3}{2} \log \left( \log x \frac{\log \log T}{\log T} \right) \right)^{5+15\delta/2} \leq (\log \log x)^{5+15\delta/2}.$$

Using these estimates, we continue the estimation of (17) with

$$\begin{aligned} & \leq \exp \left( \log \log x + (\log x)^{3/5} (\log \log x)^{3/2-3c/2+5+15\delta/2} \right) \\ & = \exp \left( (\log x)^{3/5} (\log \log x)^{13/2-3c/2+15\delta/2} + \log \log x \right) \\ & \ll_\delta \exp \left( (\log x)^{3/5} (\log \log x)^{13/2-3c/2+8\delta} \right). \end{aligned}$$

Now we resume everything including the term  $\exp \left( D \sqrt{\frac{\log x}{\log \log x}} \right)$  again, we obtain

$$B(x, \chi, q) \ll_\delta \exp \left( (\log x)^{3/5} (\log \log x)^{13/2-3c/2+8\delta} \right) \exp \left( (D+1) \sqrt{\frac{\log x}{\log \log x}} \right),$$

and using the estimate

$$\begin{aligned} & (\log x)^{3/5} (\log \log x)^{13/2-3c/2+8\delta} + (D+1) \sqrt{\frac{\log x}{\log \log x}} \\ & \ll (\log x)^{3/5} (\log \log x)^{13/2-3c/2+8\delta} \left( 1 + \log(x)^{-1/10} (\log \log x)^{3c/2} \right) \\ & \ll (\log x)^{\frac{3}{5}} (\log \log x)^{13/2-3c/2+8\delta}, \end{aligned}$$

we obtain finally

$$B(x, \chi, q) \ll_\delta \exp \left( (\log x)^{3/5} (\log \log x)^{13/2-3c/2+8\delta} \right).$$

□

Now we still have to consider the principal character mod  $q$ , for this we use the result of the zeta-function.



**Lemma 3.** *Let  $q \in \mathbb{N}$ ,  $x \geq q > 1$ , then we have for the principal character  $\chi_0 \bmod q$  the estimate*

$$A(x, \chi_0, q) \ll_{\delta} \sqrt{x} \exp \left( (\log x)^{1/2} (\log \log x)^{5/2+4\delta} \right).$$

**Proof:** Due to the formula

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left( 1 - \frac{1}{p^s} \right),$$

we use the estimate for the zeta-integral. So we estimate the product  $\left| \prod_{p|q} (1 - p^{-s})^{-1} \right|$  for  $\sigma \geq \frac{1}{2}$ .

For this, consider the logarithm of the product and include the series expansion of the logarithm:

$$\begin{aligned} \left| \sum_{p|q} -\log \left( 1 - \frac{1}{p^s} \right) \right| &= \left| \sum_{p|q} - \sum_{k \in \mathbb{N}} (-1)^{k+1} \frac{(-p^{-s})^k}{k} \right| = \left| \sum_{p|q} \sum_{k \in \mathbb{N}} (-1)^{2k+2} \frac{1}{kp^{ks}} \right| \\ &\leq \sum_{p|q} \sum_{k \in \mathbb{N}} \frac{1}{kp^{k/2}} = \sum_{p|q} \frac{1}{p^{1/2}} + \frac{1}{2} \sum_{p|q} \frac{1}{p} + \sum_{p|q} \sum_{k \geq 2} \frac{1}{kp^{k/2}} \\ &\leq \sum_{i=1}^{2 \log q} \frac{1}{p_i^{1/2}} + \frac{1}{2} \sum_{p \leq q} \frac{1}{p} + O(1) \\ &\ll \sqrt{\frac{\log q}{\log \log q}} + \log \log q + O(1). \end{aligned}$$

We conclude

$$|L(s, \chi_0)|^{-1} \ll |\zeta(s)|^{-1} \exp \left( D \sqrt{\frac{\log q}{\log \log q}} \right)$$

for an absolute constant  $D > 0$ .

Since  $\sqrt{\frac{\log q}{\log \log q}}$  is monotonic increasing in  $q$ , we have for  $x \geq q$

$$L(s, \chi_0)^{-1} \ll \zeta(s)^{-1} \exp \left( D \sqrt{\frac{\log x}{\log \log x}} \right).$$

Now the additional term  $\sqrt{\frac{\log x}{\log \log x}}$  does not disturb the magnitude of the exponent in the final result, since we have

$$\begin{aligned} \left| \int_{S(x, \chi, q)} L(z, \chi_0)^{-1} \frac{x^z}{z} dz \right| &\ll \int_{S(x, \chi, q)} \left| \zeta(z)^{-1} \frac{x^z}{z} \right| dz \exp \left( D \sqrt{\frac{\log x}{\log \log x}} \right) \\ &\ll_{\delta} \sqrt{x} \exp \left( (\log x)^{1/2} (\log \log x)^{5/2+4\delta} + D \sqrt{\frac{\log x}{\log \log x}} \right) \\ &\ll \sqrt{x} \exp \left( (\log x)^{1/2} (\log \log x)^{5/2+4\delta} \right), \end{aligned}$$

where we have set  $c = \frac{5}{2} + 3\delta$  in the estimate at the end of the paper of [5]. □

**Proof of Theorem 1:**

Let  $q > 2$ , since for  $q = 2$  there is only the principal character and we can use then the sharper result from Lemma 3.

We use equation (10), Lemma 1 and Lemma 2 and set  $c = \frac{11}{5} + \frac{16}{5}\delta$ , together with Lemma 3 we obtain

$$\begin{aligned}
\left| M(x, a, q) \right| &\leq \frac{1}{\varphi(q)} \sum_{\chi(q)} \left| \sum_{n \leq x} \chi(n) \mu(n) \right| = \frac{1}{\varphi(q)} \sum_{\chi(q)} \left| A(x, \chi, q) \right| + O(\log x) \\
&= \frac{1}{\varphi(q)} |A(x, \chi_0, q)| + \frac{1}{\varphi(q)} \sum_{\substack{\chi(q) \\ \chi \neq \chi_0}} \left| A(x, \chi, q) \right| + O(\log x) \\
&\ll_{\delta} \frac{1}{\varphi(q)} \sqrt{x} \exp \left( (\log x)^{1/2} (\log \log x)^{5/2+4\delta} \right) \\
&\quad + \frac{\varphi(q) - 1}{\varphi(q)} \sqrt{x} \exp \left( (\log x)^{3/5} (\log \log x)^{16/5+16\delta/5} \right) \\
&\ll \sqrt{x} \exp \left( (\log x)^{3/5} (\log \log x)^{16/5+16\delta/5} \right).
\end{aligned}$$

Since  $\delta \in (0, 1]$  can be chosen arbitrary, we get the assertion with the choice  $\delta = \frac{5}{16}\varepsilon$ . □

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